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Modeling cyber rumor spreading over mobile social networks: A compartment approach

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ABSTRACT

By incorporating the spreading characteristics of cyber rumors over mobile social networks, we newly develop a dynamic system to model rumor spreading dynamics by the compartment method. Specifically, a couple of a user and an attached device is viewed as a node, and all network nodes are separated into four compartments: Rumor-Neutral, Rumor-Received, Rumor-Believed and Rumor-Denied. Some transition parameters among these groups are introduced. Additionally, the role of memory, user's ability to distinguish the rumors and rumor-denier's behavior of refuting rumors are also incorporated. The stability of the equilibria of the model system is addressed, and the influence of model parameters upon the threshold is analyzed. Finally, numerical simulations illustrate the theoretical results, and also motivate us to propose suitable measures to control cyber rumor spreading by properly adjusting the parameter values.

1. Introduction

Rumors bring great harm to the social life of human beings, and their spreading is significantly affected by people's social communication [1]. The authenticity of a rumor is usually not confirmed or badly misinformed. The content of rumors can vary from traditional gossip to deliberately fabricated false information. The widespread proliferation of false rumors can lead to a series of serious hazards, such as social instability, impacts on election results or large financial losses [2–5]. Therefore, the modeling study of rumor spreading is of great importance and is always an extensive research project.

As mobile networks and applications develop rapidly, the number of network users is dramatically growing. For instance, the number of mobile phone users worldwide reached 4.43 billion in 2017 [6]. Nowadays, Internet clients mainly use the mobile communication applications for deriving breaking news or recent messages, such as Facebook, Twitter and Myspace. Meanwhile, cyber rumors are possibly spreading widely through these social networks.

An effective model for rumor spreading will have major theoretical significance in inhibiting the spreading processes, and then propose practical measures for minimizing damages caused by rumors [7,8]. Study of rumor spreading is primarily based on either models of opinion dynamics [9–11], or upon models of biological epidemics [12–15]. In the early stages, Daley and Kendall [16] studied rumor spreading and proposed the classic Daley–Kendall (DK) model, in which the population was separated into three categories: ignorants (people who are unaware of the rumor), spreaders (people who spread the rumor), and stiflers (people who are aware of the rumor but choose not to spread it). Considering that rumors spread

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through direct contact between spreaders and others, Maki and Thompson [17] later developed the MK model by modifying the DK model. These models along with their variants have been applied extensively in the modeling study of rumor spreading [18,19]. Based on information entropy, Wang et al. [20] presented a comprehensive model which incorporates conformity effects and variations in the degree of trust. Zhao et al. [21] studied rumor spreading dynamics on homogeneous networks incorporating the forgetting mechanism, and concluded that the final state of stiflers depended greatly on the average degree of networks. With the development of Internet and computer networks [22,23], rumor propagation dynamics also incorporated the structure topology of social networks [24,25]. By considering the relationship between messages, Wang and Zhao [26] proposed an extended SIS model for multi-messages spreading over complex networks. The cyber rumor spreading behavior has also a little similarity to malware prevalence [27,28] and the strategy diffusion within the population [29,30], and readers can refer to Refs. [31–35] for more information.

The studies about rumor spreading mentioned above mainly considered the mechanism of rumor diffusion among people (rumor spreading from person to person), not incorporating the influence of interactions between users and mobile devices on cyber rumor propagation. The methods in which rumors spread are diverse and continuously evolving, from the traditional way of mouth-to-mouth to the current popular ways of social networks such as microblogs and emails. Social communication applications have many characteristics of which the most apparent characteristic is quick transmission among users. Therefore, although the previous models contribute much to the study of rumor spreading, there are still some flaws when they are directly applied to model cyber rumor spreading by new mechanisms. For example, technological applications can affect the mechanisms of rumor spreading. Nowadays, people get the latest news mainly through social network platforms and spread the messages by direct forwarding. Thus, it is worth studying how the unique mechanisms of mobile social applications affect cyber rumor spreading. In this paper, we are devoted to developing a novel dynamical model by taking into account the impact of social media applications and the rumor-refuting actions of rumor-deniers. The properties of propagation dynamics of the model, especially the global stability of the equilibria, are addressed. Some numerical simulations are also designed to analyze the model with some specific parameters. Finally, some suggestions are put forward for people to control the spreading of cyber rumor.

2. Model description

Unlike epidemic diseases, cyber rumors can proliferate through social networks or communication platforms such as social media applications. Next, a new dynamic model will be developed to address the impacts of some key factors on cyber rumor spreading. Following the dynamic analysis, we further discuss how to effectively control the propagation of cyber rumors.

The rapid development of network technology, including 4G/5G and mobile internet, leads to the dramatic increase in network services. In the traditional rumor propagation models, nodes simply represent the people since the rumors directly spread among them through word-of-mouth. However, the situation for cyber rumor propagation is much more complicated. The most prominent difference is the diversification of communication mechanisms for online rumors. Specifically, network users are now used to derive timely information from their mobile devices such as smartphones and tablets.

By considering the unique mechanism of cyber rumor spreading over mobile networks, we innovatively put forward that the nodes of the propagating network for online rumors are couples each of which consists of a user and an attached mobile device. Users get information by reading the messages (rumors) from the SNS (Social Network Service) applications installed on their devices.

We consider that all applications have a buffer memory which is similar to the cache of the computer, it can receive the news when people cannot read the new message immediately. However, the number of messages is vast, users are impossible to immediately read every message received from the social media. Based on this physical consideration, we introduce a new state called Received status to depict the situation that the rumor messages on a device have not noticed by users.

Thus, in our model, all the network nodes can be divided into four groups (see Fig. 1).

- Rumor-Neutral nodes (U-nodes). The social accounts of users are vulnerable to receive the rumor information at any time. These nodes mean that the devices have either not yet received the propagating rumor, or have previously received it but have since forgotten it and become neutral again. $U(t)$ is introduced to denote the quantity of U-nodes at time t, i.e., the number of devices (nodes) having not received the rumor message.
- Rumor-Received nodes (R-nodes). The devices or accounts of these nodes have already received the rumor information, but the corresponding users have not read them. Usually for the unread messages, social platforms have certain caching mechanisms which will remind users to read once they become online. $R(t)$ is used to denote the quantity of R-nodes at time t, i.e., the number of devices (nodes) having already received the rumor message (but not read yet by users).
- Rumor-Believed nodes (B-nodes). The users of these nodes have read the rumor information through their devices and believe that the information is true, even choosing to share the rumor information to their friends by forwarding the messages via the network. $B(t)$ is introduced to represent the quantity of B-nodes at time t, i.e., the number of nodes whose users choose to believe it after reading the rumor message through their devices at time t.
- Rumor-Denied nodes (D-nodes). After reading the rumor information, the users of these nodes think that it is false. Thus, they choose to deny it and also possibly tell others that it is a false rumor. $D(t)$ is introduced to denote the quantity of

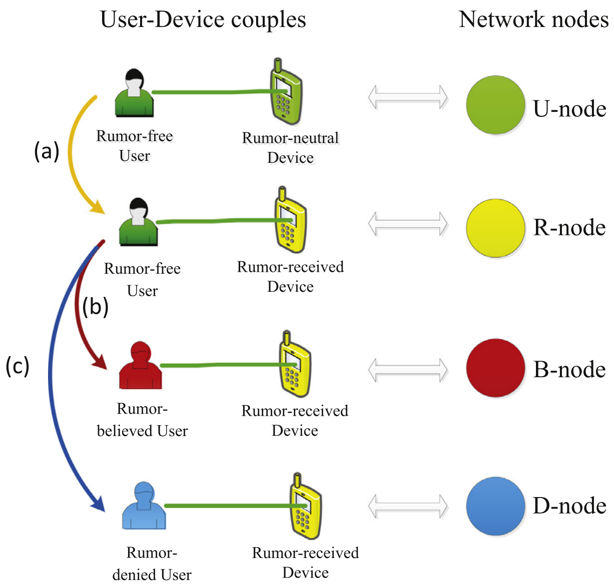


Fig. 1. Illustration of the compartmental nodes. (a) A rumor-neural device will become rumor-received if some users in its friend list forward the rumor message. (b) After reading the rumor message from the mobile device, the user believes in the content. (c) The user thinks the rumor is false, and will not forward.

D-nodes at time *t*, i.e., the number of nodes whose users have read the rumor message and then choose to deny it at time *t*.

Remark 1. As defined above, the U-nodes include two cases: one is that the devices do not receive rumor messages (obviously unknown to users too), the other is that users have forgotten the rumors (for this case, the devices received rumors previously, but can be viewed as rumor-unreceived recently). For both cases, we can think that the U-nodes are free to cyber rumors. However, for R-nodes the users are possible to get the rumor information from their devices. Moreover, we think that the number of users knowing the rumor messages will grow with the increasing number of R-nodes. Thus, it is very significant to pay attention to the group of R-nodes, and in our model the U-nodes and R-nodes are separately introduced. By this U-node and R-node setup, we also expect to proceed some future works. For example, some controlling strategies or practical technique measures can be taken to reduce the number of R-nodes by prohibiting cyber rumor spreading.

For simplicity, here we do not consider that some people are outside certain rumor networks and will never receive rumor messages on SNSs. In this work, we just aim to model cyber rumor spreading over a fixed connected social network. That is, only the nodes of a closed propagating network are considered. Then, each rumor-neutral node is possible to receive rumor messages at some time.

Next, we will present some other hypotheses and parameters.

(A1) A U-node is assumed to successfully receive the rumor message sent by a single B-node with the probability of β_0 per unit time.

At each time step, rumor spreading occurs between B-nodes and U-nodes that are connected. The users who believe the rumor information are possible to forward the messages over their social networks, leading to rumor prevalence. Upon receiving the rumor message, a U-node is considered to immediately convert to be a R-node. Consider a specific U-node with a number of m connected B-nodes, then the single U-node can turn to a R-node by the probability of $1-(1-\beta_0)^m$ within a unit time.

Remark 2. Note that we actually make another potential assumption here that all individuals over a propagating network have equal influence so that we derive the above uniform probability. However, the actual case is that some users can be more influential than others in a person's SNS, e.g., if an individual receives rumor messages from trusted friends versus someone else not known very well. For this kind of general cases, we will address them in our potential future work, such as considering a patchy/group model with varying influencers.

(A2) An R-node is assumed to possibly transform to be a B-node with the probability of ϕ_b or to be a D-node with the probability of ϕ_d per unit time.

After reading the rumor message, a user will usually make a judgment to believe in it or to deny it. This process may be affected by many factors such as users' educational level and user activeness. The education level of users is an important

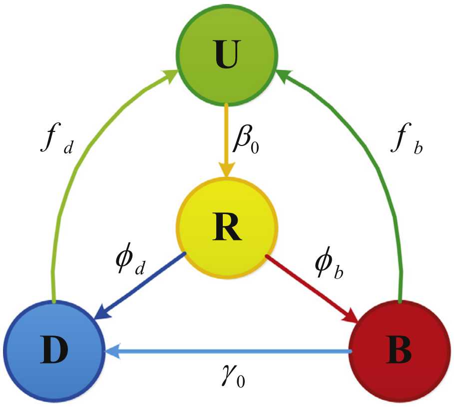


Fig. 2. General schematic of the new URBD model with variant transform rates among the states.

fact which affects the probability on rumor recognition. Considering the user's education variance, the users of high education level will recognize the rumor with high probability and turn into a rumor-denied user. Meanwhile, the R-node users with lower education level are likely to fail to recognize the rumor content and thus transform to be a rumor-believed user. There also exist other situations. For example, if users remember that they had already read similar messages (already made the judgement), then they can quickly become rumor-denied nodes or rumor-believed nodes. On the contrary, if the users have read the rumor message many times but without knowing it is false, then the probability of converting to be a rumor-believed user will become higher. Here, for simplicity, we consider that ϕ_b and ϕ_d are constants, and $\phi_b > 0$, $\phi_d > 0$ with $\phi_b + \phi_d \le 1$. However, there is also a possible situation that an R-node can remain an R-node and not filter into D-nodes or B-nodes within a period of time, since the users possibly miss the rumor messages on their devices or are not interested in the rumor content after reading them. Thus, in fact, the value of $\phi_b + \phi_d$ should be less than 1, i.e., the state of an R-node keeping unchanged with the probability of $1 - \phi_b - \phi_d$ per unit time.

(A3) It is assumed that a B-node and a D-node will both convert to be a U-node with certain probabilities, respectively.

The rates f_b and f_d are introduced to depict the probabilities that a B-node and a D-node change back to a U-node per unit time, respectively. This may occur because of some reasons. For example, it is a common phenomenon that people probably forget something after a certain period of time. Based on this consideration, deniers and believers possibly forget the rumor information and then become free to rumors again (i.e., D-nodes and B-nodes turning to be U-nodes). Thus, the forgetting characteristics of users can be an important factor to influence the state transition and affect the spreading dynamics of cyber rumors. In our model, we consider that both f_b and f_d are positive.

(A4) It is assumed that a rumor-believed node will turn to be a rumor-denied node with the probability of γ_0 per unit time by the influence of a single neighboring rumor-denied node.

Once cyber rumors are found spreading over social networks, the government (related agencies) or users will take certain countermeasures to prevent their diffusion. Here we will focus on the influence of user behavior on rumor inhibition process, and thus we just consider the mechanism of refuting rumors that rumor-deniers will probably release or forward information on refuting the rumor through social networks. Specifically, after receiving the rumor-refuting message sent by one rumor-denied user, a rumor-believed user will change the previous cognition of the rumor content, turning to be a rumor-denied user with probability γ_0.

Based on the above considerations, cyber rumor spreading leads to dynamic transitions among these states, shown in Fig. 2. We also denote $N(t)$ by the number of all nodes at time t. For simplicity, we first consider that cyber rumors are spreading over a fully-connected static network. In this case, $N(t)$ would be a constant all the time in our model. Then, by certain calculations, we can get the dynamic differential model as follows

$$\begin{cases} \dfrac{dU(t)}{dt} = f_b B(t) + f_d D(t) - [1 - (1-\beta_0)^{B(t)}]U(t), \\ \dfrac{dR(t)}{dt} = [1 - (1-\beta_0)^{B(t)}]U(t) - \phi_d R(t) - \phi_b R(t), \\ \dfrac{dD(t)}{dt} = \phi_d R(t) - f_d D(t) + [1 - (1-\gamma_0)^{D(t)}]B(t), \\ \dfrac{dB(t)}{dt} = \phi_b R(t) - f_b B(t) - [1 - (1-\gamma_0)^{D(t)}]B(t), \end{cases} \tag{1}$$

where f_b, f_d, β_0, ϕ_d, ϕ_b, γ_0 are positive parameters introduced above.

Remark 3. Note that the model (1) is formulated over a fully-connected network. Thus, theoretically, it is only suitable to model rumor spreading on complete propagating networks. However, most of real networks are not fully connected. Therefore, in order to model cyber rumor spreading over a wider class of networks, we need to make some revisions on system (1).

Denote $h(x, y) = 1 - (1-x)^y, x \in [0, 1], y \geq 0,$ where $x \in \{\beta_0,\ \gamma_0\}$, and y can be $B(t)$ or $D(t)$, respectively. By the Taylor expansion with respect to the variable x, we have $h(x, y) = h(0, y) + h'(0, y)(x-0) + o[(x-0)] = yx + o(x)$. Here, we make an approximation $h(x,\ y) \approx xy$, then system (1) can be simplified as the following new model

$$\begin{cases} \dfrac{dU(t)}{dt} = f_b B(t) + f_d D(t) - \beta_0 B(t)U(t), \\ \dfrac{dR(t)}{dt} = \beta_0 B(t)U(t) - \phi_d R(t) - \phi_b R(t), \\ \dfrac{dD(t)}{dt} = \phi_d R(t) - f_d D(t) + \gamma_0 D(t)B(t), \\ \dfrac{dB(t)}{dt} = \phi_b R(t) - f_b B(t) - \gamma_0 D(t)B(t). \end{cases} \tag{2}$$

The model (2) can be also derived by the traditional mean-field theory with homogeneous mixing. Note that system (2) and system (1) are two different models.

The topological characteristics of real spreading networks are much more complicated. Even for homogeneous networks, they are varying depending on different network sizes and distinct node degrees. To make the above model applicable to study rumor spreading over a broader class of homogeneously-mixed networks, here we extend it by introducing a parameter $\delta \in (0,\ 1]$ to roughly describe the percentage of nodes adjacent to a node compared to the fully-connected network. For example, the δ for a specific homogeneous network with average node degree of κ can be generally defined as $\delta = \kappa/(N-1)$ (just an example, maybe not adequate). Then, we can get the following model

$$\begin{cases} \dfrac{dU(t)}{dt} = f_b B(t) + f_d D(t) - \beta_0 \delta B(t)U(t), \\ \dfrac{dR(t)}{dt} = \beta_0 \delta B(t)U(t) - \phi_d R(t) - \phi_b R(t), \\ \dfrac{dD(t)}{dt} = \phi_d R(t) - f_d D(t) + \gamma_0 \delta D(t)B(t), \\ \dfrac{dB(t)}{dt} = \phi_b R(t) - f_b B(t) - \gamma_0 \delta D(t)B(t). \end{cases} \tag{3}$$

Note that we just formulate the model (3) incorporating the parameter δ which is not well-defined yet, and it is worth studying how to properly define the parameter δ. Here, we propose the study of adequate definitions of δ and deep analysis of model (3) as a future direction.

In the sequel, we simplify the formulation of model (3) by denoting $\beta = \beta_0 \delta N$ and $\gamma = \gamma_0 \delta N$, and then analyze its dynamics. Since the spreading network is considered to be a static one, the number of nodes over the social network is fixed. Thus, the variables in model (3) can be normalized by setting $u(t) = U(t)/N, r(t) = R(t)/N, b(t) = B(t)/N$, and $d(t) = D(t)/N$, where N denotes the total number of network nodes. Then model (3) has the following simpler form

$$\begin{cases} u'(t) = f_b b(t) + f_d d(t) - \beta b(t)u(t), \\ r'(t) = \beta b(t)u(t) - \phi_d r(t) - \phi_b r(t), \\ d'(t) = \phi_d r(t) - f_d d(t) + \gamma d(t)b(t), \\ b'(t) = \phi_b r(t) - f_b b(t) - \gamma d(t)b(t). \end{cases} \tag{4}$$

By the identity $u(t) + r(t) + d(t) + b(t) = 1,$ the above model can be equivalently reduced to the following system of three differential equations

$$\begin{cases} r'(t) = \beta b(t)(1 - r(t) - d(t) - b(t)) - (\phi_d + \phi_b)r(t), \\ d'(t) = \phi_d r(t) - f_d d(t) + \gamma d(t)b(t), \\ b'(t) = \phi_b r(t) - f_b b(t) - \gamma d(t)b(t). \end{cases} \tag{5}$$

Note that the meaningful physical domain for system (5) is $\Omega = \{(x, y, z) \in \mathbb{R}^3_+ | x + y + z \leq 1\}$. Since model (4) is equivalent to model (5) in the sense of $u(t) + r(t) + d(t) + b(t) = 1,$ it suffices to analyze system (5) in the sequel. Next, the following useful lemma is necessary.

Lemma 1. *The set Ω is positively invariant to system (5).*

Proof. Denote $\mathbf{x}(t) = (r(t), d(t), b(t))^{\mathrm{T}}$ and then system (5) can be rewritten as

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)),$$

where $\mathbf{f}(\mathbf{x}) = (\beta x_3(1-x_1-x_2-x_3)-(\phi_d+\phi_b)x_1, \phi_d x_1 - f_d x_2 + \gamma x_2 x_3, \phi_b x_1 - f_b x_3 - \gamma x_2 x_3)^{\mathrm{T}}$. Note that Ω is obviously a compact set. We only need to prove that if $\mathbf{x}(0)\in\Omega$, then $\mathbf{x}(t)\in\Omega$ for all $t\geq 0$. Note that $\partial\Omega$ consists of three plane segments:

$$\begin{aligned}
P_1 &= \{(x,y,0)|x,y\in[0,1], x+y\leq 1\},\\
P_2 &= \{(x,0,z)|x,z\in[0,1], x+z\leq 1\},\\
P_3 &= \{(0,y,z)|y,z\in[0,1], y+z\leq 1\},\\
P_4 &= \{(x,y,z)\in\mathbb{R}^3_+|x+y+z=1\},
\end{aligned}$$

which have $\mathbf{v}_1=(0,0,-1), \mathbf{v}_2=(0,-1,0), \mathbf{v}_3=(-1,0,0), \mathbf{v}_4=(1,1,1)$ as their outer normal vectors, respectively. Let $\mathbf{p}=(p_1,\ldots,p_n)$ and $\mathbf{q}=(q_1,\ldots,q_n)$ be two vectors, and here we denote $\langle\mathbf{p},\mathbf{q}\rangle$ as the scalar product of vectors $\mathbf{p}$ and $\mathbf{q}$, i.e., $\langle\mathbf{p},\mathbf{q}\rangle = p_1q_1+\cdots+p_nq_n$. Through certain calculations, we have

$$\begin{aligned}
\left\langle \frac{d\mathbf{x}(t)}{dt}|_{\mathbf{x}\in P_1}, \mathbf{v}_1\right\rangle &= -\phi_b x_1 = -\phi_b r(t)\leq 0,\\
\left\langle \frac{d\mathbf{x}(t)}{dt}|_{\mathbf{x}\in P_2}, \mathbf{v}_2\right\rangle &= -\phi_d x_1 = -\phi_d r(t)\leq 0,\\
\left\langle \frac{d\mathbf{x}(t)}{dt}|_{\mathbf{x}\in P_3}, \mathbf{v}_3\right\rangle &= -\beta x_3(1-x_2-x_3) = -\beta b(t)(1-d(t)-b(t))\leq 0,\\
\left\langle \frac{d\mathbf{x}(t)}{dt}|_{\mathbf{x}\in P_4}, \mathbf{v}_4\right\rangle &= -f_d x_2 - f_b x_3 = -f_d d(t) - f_b b(t)\leq 0.
\end{aligned}$$

Thus, the claimed result follows directly by Lemma 3.2 in [27] (see also Ref. [36]). The proof is complete. □

By applying the method proposed by Driessche and Warmough [37], the threshold of model (5) can be calculated as (see **Appendix** for details)

$$R_0 = \frac{\phi_b\beta}{f_b(\phi_b+\phi_d)}. \tag{6}$$

In many epidemiological models, often depicted by dynamical systems, the thresholds are usually called as *basic reproduction numbers* which essentially mean the number of infectious cases that one case generates on average over the course of the disease's infectious period. Similarly, the above threshold of model (5) for cyber rumor spreading can be explained as the average number of secondary R-nodes produced by a single B-node through forwarding the rumor message.

Remark 4. Although we previously clarify that the parameter f_b is positive, it seems reasonable to consider the case $f_b=0$ physically since B-nodes filter to D-nodes (via γ_0) and back to U-nodes (via f_d). Note that the threshold R_0 is not defined for $f_b=0$. For this case, the matrix $V=\begin{pmatrix} f_b & 0\\ 0 & \phi_b+\phi_d\end{pmatrix}$ in **Appendix** happens to be singular, i.e., its inverse is not well defined. Thus, we cannot calculate the threshold by the approach in **Appendix** for the special case $f_b=0$. For calculating the threshold of system (5) with $f_b=0$, we propose it as an open question here and we guess that it can be possibly calculated by the physical meanings of the parameters (see the calculating processes for the threshold of the compartment model in [27]).

3. Global stability of the equilibria

Studying the stability properties of model systems is of great significance, e.g., providing insights for making proper controlling measures [38–41]. Thus, this section aims to fully analyze the dynamic properties of system (5), including the equilibria and their global stability. It is easy to see that system (5) always possesses a unique rumor-free equilibrium $E_0=(0,0,0)$, which is a state corresponding to the absence of cyber rumor. On the other hand, we define a rumor equilibrium as a state of which the B-component is positive, which means that the cyber rumor will keep spreading over the networks. The following lemma proves the existence of a rumor equilibrium under some conditions.

Lemma 2. *Consider system (5), and denote*

$$\Phi = \frac{[(\gamma-f_d)\phi_b\beta - \gamma f_b(\phi_b+2\phi_d) - R_0 f_b^2\phi_d] + \sqrt{\Delta}}{2[\gamma\phi_b\beta+\gamma^2(\phi_b+\phi_d)+R_0 f_b f_d\gamma]},$$

where

$$\Delta = [(\gamma-f_d)\phi_b\beta - \gamma f_b(\phi_b+2\phi_d) - R_0 f_b^2\phi_d]^2 + 4[\gamma\phi_b\beta+\gamma^2(\phi_b+\phi_d)+R_0 f_b f_d\gamma] f_b^2\phi_d(R_0-1).$$

Then, the following assertions hold.

1. *There exists no rumor equilibrium if* $R_0 \le 1$.
2. *If* $R_0 > 1$ *and* $\Phi \ge \dfrac{f_b(R_0-1)}{f_bR_0+\gamma}$, *then there exists no rumor equilibrium.*
3. *If* $R_0 > 1$ *and* $\Phi < \dfrac{f_b(R_0-1)}{f_bR_0+\gamma}$, *then there exists a unique rumor equilibrium* $E^* = (r^*, d^*, b^*)$, *where*
$$d^* = \Phi,$$
$$b^* = \frac{\phi_b\beta(1-d^*) - (f+\gamma d^*)(\phi_b+\phi_d)}{(\phi_b+f+\gamma d^*)\beta},$$
$$r^* = \frac{\beta b^*(1-b^*-d^*)}{\phi_b+\phi_d+\beta b^*}.$$

Proof. Let $E^* = (r^*, d^*, b^*)$, where $r^*, d^*, b^* \in [0, 1]$, be an equilibrium of system (5). Then, to get its explicit form, we need to solve the following system of three equations

$$\begin{cases} \beta b^*(1-r^*-d^*-b^*) - (\phi_d+\phi_b)r^* = 0, \\ \phi_d r^* - f_d d^* + \gamma d^* b^* = 0, \\ \phi_b r^* - f_b b^* - \gamma d^* b^* = 0. \end{cases} \tag{7}$$

From the first equation of (7), we can get

$$r^* = \frac{\beta b^*(1-b^*-d^*)}{\phi_b+\phi_d+\beta b^*}. \tag{8}$$

By substituting (8) into the third equation of (7), we can get the following equation

$$b^*[\phi_b\beta(1-b^*-d^*) - (f_b+\gamma d^*)(\phi_b+\phi_d+\beta b^*)] = 0. \tag{9}$$

From the above Eq. (9), we can get that $b^* = 0$, or

$$b^* = \frac{\phi_b\beta(1-d^*) - (f_b+\gamma d^*)(\phi_b+\phi_d)}{(\phi_b+f_b+\gamma d^*)\beta} = \frac{\phi_b[(R_0-1)f_b - (R_0f_b+\gamma)d^*]}{(\phi_b+f_b+\gamma d^*)f_bR_0}. \tag{10}$$

The case $b^* = 0$ immediately leads to the unique rumor-free equilibrium $E_0 = (0, 0, 0)$ of system (5). For the other case, if $R_0 \le 1$, then b^* is obviously nonpositive, implying that there is no rumor equilibrium for this case. Therefore, the first assertion holds.

Next, we will consider the other case $R_0 > 1$. Substituting the expression of b^* (identity (10)) into $1 - b^* - d^*$ yields that

$$1 - b^* - d^* = \frac{(f_b+\gamma d^*)(\phi_b+\phi_d+\beta(1-d^*))}{(\phi_b+f_b+\gamma d^*)\beta}.$$

It follows by substituting the expression of r^* (identity (8)) into the second equation of (7) that

$$\begin{aligned} &\phi_d\beta b^*(1-b^*-d^*) + d^*(\gamma b^* - f_d)(\phi_b+\phi_d+\beta b^*) = 0 \\ \Leftrightarrow\ &\beta[\phi_d b^* + d^*(f_d-\gamma b^*)](1-b^*-d^*) + d^*(\gamma b^* - f_d)[\phi_b+\phi_d+\beta(1-d^*)] = 0. \end{aligned}$$

By substituting the expression of $1 - b^* - d^*$ into the above identity, we can get the following equation

$$[\gamma(\phi_b+\phi_d)b^*d^* + \phi_d f_b b^* - \phi_b f_d d^*] \times [\phi_b+\phi_d+\beta(1-d^*)] = 0.$$

Note that the above identity can be seen as a cubic equation of the variable d^*, and it can be obviously split into a quadratic part and a linear part. Solving the linear equation $\phi_b+\phi_d+\beta(1-d^*) = 0$, we can immediately get that $d^* = 1 + \frac{\phi_b+\phi_d}{\beta} > 1$, which obviously contradicts to the requirement $d^* \le 1$. Thus, this case must be neglected.

For the other quadratic equation, it can be further simplified as

$$\begin{aligned} &\gamma(\phi_b+\phi_d)b^*d^* + \phi_d f_b b^* - \phi_b f_d d^* = 0 \\ &\Leftrightarrow \gamma\phi_b\beta(\phi_b+\phi_d)d^*(1-d^*) - \gamma(\phi_b+\phi_d)^2 d^*(f_b+\gamma d^*) \\ &\quad +\phi_b\phi_d f_b\beta(1-d^*) - \phi_d f_b(\phi_b+\phi_d)(f_b+\gamma d^*) - \phi_b f_d\beta d^*(\phi_b+f_b+\gamma d^*) = 0 \\ &\Leftrightarrow -[\gamma\phi_b\beta(\phi_b+\phi_d) + \gamma^2(\phi_b+\phi_d)^2 + \phi_b f_d\beta\gamma](d^*)^2 \\ &\quad +[(\gamma-f_d)\phi_b\beta(\phi_b+\phi_d) - \gamma f_b(\phi_b+\phi_d)(\phi_b+2\phi_d) - \phi_b\phi_d\beta f_b]d^* + \phi_b\phi_d f_b\beta(1-1/R_0) = 0 \\ &\Leftrightarrow [\gamma\phi_b\beta + \gamma^2(\phi_b+\phi_d) + R_0 f_b f_d\gamma](d^*)^2 - [(\gamma-f_d)\phi_b\beta - \gamma f_b(\phi_b+2\phi_d) - R_0 f_b^2\phi_d]d^* - f_b^2\phi_d(R_0-1) = 0. \end{aligned}$$

Easily, we know that the above quadratic equation of d^* has two real roots with different signs, and the positive one is $d^* = \Phi$.

Following the expression of (10), it can be easily seen that b^* may be positive or nonpositive for the case $R_0 > 1$. Thus, we proceed by discussing two subcases.

Case 1: If $R_0 > 1$, and $\Phi \ge \dfrac{f_b(R_0-1)}{f_bR_0+\gamma}$, then we can get $b^* \le 0$ since $d^* = \Phi$. This indicates that there exists no rumor equilibrium for this case. Therefore, the second assertion holds.

Case 2: If $R_0 > 1$, and $\Phi < \frac{f_b(R_0-1)}{f_bR_0+\gamma}$, then it follows by (10) and $d^* = \Phi$ that b^* is positive, implying that a unique rumor equilibrium exists. Finally, we can get the value of r^* by substituting the values of d^* and b^* into (8). Thereby, the third assertion holds.

The proof is complete. □

Remark 5. In many similar epidemic models, the unique existence of the endemic equilibrium can be easily checked when the threshold is greater than one. But, for the model (5), this seems uneasy. Lemma 2 shows that there are two subcases when $R_0 > 1$, i.e., there exists no rumor equilibrium if $\Phi \geq f_b(R_0-1)/(f_bR_0+\gamma)$, and there exists a unique rumor equilibrium if $\Phi < f_b(R_0-1)/(f_bR_0+\gamma)$. By intuition and some numerical simulations (see Fig. 6), we guess that the inequality $\Phi < f_b(R_0-1)/(f_bR_0+\gamma)$ may always hold for all defined parameter choices satisfying $R_0 > 1$. If so, then the second assertion of Lemma 2 can be deleted. Unfortunately, since the expression of Φ is much complicated, we cannot theoretically confirm it in spite of paying some efforts.

Next, we will first present a theorem to address the global stability of the rumor-free equilibrium of system (5).

Theorem 1. *If $R_0 < 1$, then the rumor-free equilibrium point E_0 of system (5) is globally asymptotically stable with respect to Ω.*

Proof. We proceed by using the Lyapunov direct method with undetermined coefficients. Consider the following candidate function

$$V(r(t), b(t), d(t)) = r(t) + \omega_2 d(t) + \omega_1 b(t),$$

where ω_1 and ω_2 are positive constants to be determined. Clearly, it follows by $r(t) \geq 0$, $b(t) \geq 0$, $d(t) \geq 0$ that $V \geq 0$ and $V = 0$ if and only if $(r(t), d(t), b(t)) = E_0$ with respect to Ω. That is, V is positive definite. The time derivative of V along an orbit of system (5) is

$$\begin{aligned}
\frac{dV}{dt}|_{(5)} &= r'(t) + \omega_2 d'(t) + \omega_1 b'(t) \\
&= \beta b(t)(1-r(t)-b(t)-d(t)) - (\phi_b+\phi_d)r(t) + \omega_1(\phi_b r(t) - f_b b(t) - \gamma b(t)d(t)) \\
&\quad + \omega_2(\phi_d r(t) - f_d d(t) + \gamma b(t)d(t)) \\
&= \beta b(t) - \beta b(t)r(t) - \beta b^2(t) - \beta b(t)d(t) - (\phi_b+\phi_d)r(t) + \omega_1\phi_b r(t) - \omega_1 f_b b(t) - \omega_1\gamma b(t)d(t) \\
&\quad + \omega_2\phi_d r(t) - \omega_2 f_d d(t) + \omega_2\gamma b(t)d(t) \\
&= (\beta - \omega_1 f_b)b(t) + (\omega_2\gamma - \omega_1\gamma - \beta)b(t)d(t) + (\omega_1\phi_b + \omega_2\phi_d - (\phi_b+\phi_d))r(t) - \beta b(t)r(t) \\
&\quad - \beta b^2(t) - \omega_2 f_d d(t).
\end{aligned}$$

Let $\omega_1 = \beta/f_b$, and choose a suitable positive value of ω_2 such that

$$\omega_2 < \min\left\{\frac{\beta(f_b+\gamma)}{f_b\gamma}, \frac{\phi_b\beta}{\phi_d f_b}\left(\frac{1}{R_0}-1\right)\right\}.$$

Then it follows by $\omega_1 = \beta/f_b$ that $\beta - \omega_1 f_b = 0$. Since $\omega_2 < \frac{\beta(f_b+\gamma)}{f_b\gamma}$, then $\omega_2\gamma - \omega_1\gamma - \beta < 0$. Similarly, it follows by $\omega_2 < \frac{\phi_b\beta}{\phi_d f_b}(\frac{1}{R_0}-1)$ and $R_0 < 1$ that $\omega_1\phi_b + \omega_2\phi_d - (\phi_b+\phi_d) < 0$.

It can be easily concluded that $\dot{V} \leq 0$ holds for all $(r(t), d(t), b(t)) \in \Omega$. Furthermore, it can be verified that $\dot{V} = 0$ if and only if $(r(t), d(t), b(t)) = E_0$ with respect to Ω. That is, $\dot{V}$ is negative definite. Therefore, E_0 is globally asymptotically stable with respect to Ω if $R_0 < 1$. The proof is complete. □

Next, we are going to discuss the stability of the rumor equilibrium of system (5).

Theorem 2. *If $R_0 > 1$ and $\Phi < \frac{f_b(R_0-1)}{f_bR_0+\gamma}$, then the rumor equilibrium of system (5) is asymptotically stable.*

Proof. The Jacobian matrix at the states of the system (5) is shown as follows

$$J = \begin{bmatrix} -\beta b(t) - (\phi_d+\phi_b) & -\beta b(t) & -2\beta b(t) \\ \phi_d & -f_d + \gamma b(t) & \gamma d(t) \\ \phi_b & -\gamma b(t) & -f_b - \gamma d(t) \end{bmatrix}$$
$$\Rightarrow J|E^* = \begin{bmatrix} -\beta b^* - (\phi_d+\phi_b) & -\beta b^* & -2\beta b^* \\ \phi_d & -f_d + \gamma b^* & \gamma d^* \\ \phi_b & -\gamma b^* & -f_b - \gamma d^* \end{bmatrix}$$

Denote $\mathbf{I}_{n\times n}$ as the unit square matrix of size n. The characteristic equation of the above Jacobian matrix is calculated as follows

$$\begin{aligned}\det(\lambda\mathbf{I}_{3\times3}-J|E^*) &= -\begin{vmatrix}-\beta b^*-(\phi_d+\phi_b)-\lambda & -\beta b^* & -2\beta b^*\\ \phi_d & -f_d+\gamma b^*-\lambda & \gamma d^*\\ \phi_b & -\gamma b^* & -f_b-\gamma d^*-\lambda\end{vmatrix}\\ &= \lambda^3+(f_b+f_d-\gamma b^*+\gamma d^*+\beta b^*+\phi_d+\phi_b)\lambda^2\\ &\quad+[f_bf_d+f_d\gamma d^*-f_b\gamma b^*+\beta b^*(\phi_d+2\phi_b)+(\beta b^*+\phi_b+\phi_d)(f_b+f_d-\gamma b^*+\gamma d^*)]\lambda\\ &\quad+\phi_d\beta b^*(f_b+\gamma d^*-2\gamma b^*)+\phi_b\beta b^*(2f_d+\gamma d^*-2\gamma b^*)\\ &\quad+(\beta b^*+\phi_b+\phi_d)(f_df_b+f_d\gamma d^*-\gamma f_bb^*)=0.\end{aligned}$$

Next, we denote

$$\begin{aligned}\zeta_2 &= f_b+f_d-\gamma b^*+\gamma d^*+\beta b^*+\phi_d+\phi_b,\\ \zeta_1 &= [f_d\gamma d^*+f_b(f_d-\gamma b^*)+\beta b^*(\phi_d+2\phi_b)+(\beta b^*+\phi_b+\phi_d)(f_b+f_d-\gamma b^*+\gamma d^*)],\\ \zeta_0 &= \phi_d\beta b^*(f_b+\gamma d^*-2\gamma b^*)+\phi_b\beta b^*(2f_d+\gamma d^*-2\gamma b^*)+(\beta b^*+\phi_b+\phi_d)(f_df_b+f_d\gamma d^*-\gamma f_bb^*).\end{aligned}$$

It follows by the second equation of (7) that $f_d-\gamma b^*=\phi_d\gamma^*/d^*$, which immediately leads to that both ζ_2 and ζ_1 are obviously positive. Similarly, by certain calculations, we can verify that $\zeta_0>0$ and $\zeta_2\zeta_1>\zeta_0$ by applying the equations of (7) as well as the conditions $R_0>1$ and $\Phi<f_b(R_0-1)/(f_bR_0+\gamma)$. Therefore, according to the Routh–Hurwitz criterion, we can get that all the roots of the above characteristic equation have negative real parts. Thus, the rumor equilibrium is asymptotically stable. The proof is complete. □

In the following theorem, some sufficient conditions are presented for the global stability of the rumor equilibrium of system (5).

Theorem 3. *If the parameters of model (5) satisfy the requirements* $R_0>1$, $\Phi<\dfrac{f_b(R_0-1)}{f_bR_0+\gamma}$, $2b^*+r^*+d^*>1+\dfrac{\phi_b}{f_b}r^*$, *and*

$$\frac{\beta}{4}+x\gamma+\frac{x^2\phi_d^2}{4(\phi_b+\phi_d)}<xf_d, \tag{11}$$

where $x=\beta b^*(2b^*+r^*+d^*-1)/(\phi_bd^*)$, *then the rumor equilibrium of system (5) is globally asymptotically stable with respect to* $\Omega-\{E_0\}$.

Proof. The two conditions $R_0>1$ and $\Phi<\dfrac{f_b(R_0-1)}{f_bR_0+\gamma}$ guarantee that there exists a rumor equilibrium for model (5). Then, we first construct the following candidate Lyapunov function

$$L(r(t),b(t),d(t))=\frac{1}{2}\Big[(r(t)-r^*)^2+\eta_1(d(t)-d^*)^2+\eta_2(b(t)-b^*)^2\Big],$$

where η_1 and η_2 are positive constants to be determined. Obviously, $L(r(t), b(t), d(t))\geq0$ always holds and $L=0$ if and only if $(r(t),d(t),b(t))=E^*$. That is, L is positive definite. By applying the identities of (7), we can get

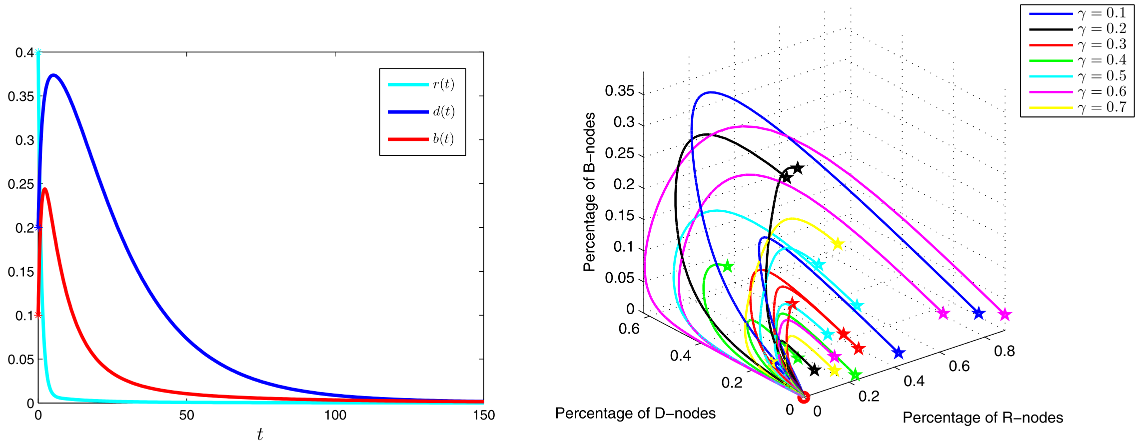
$$\begin{aligned}r'(t) &= \beta(b(t)-b^*)[1-r^*-d^*-b^*-(r(t)-r^*)-(d(t)-d^*)-(b(t)-b^*)]\\ &\quad-\beta b^*[(r(t)-r^*)+(d(t)-d^*)+(b(t)-b^*)]-(\phi_d+\phi_b)(r(t)-r^*),\\ d'(t) &= \phi_d(r(t)-r^*)-f_d(d(t)-d^*)+\gamma(d(t)-d^*)(b(t)-b^*)+\gamma b^*(d(t)-d^*)+\gamma d^*(b(t)-b^*),\\ b'(t) &= \phi_b(r(t)-r^*)-(f_b+\gamma d^*)(b(t)-b^*)-\gamma(d(t)-d^*)(b(t)-b^*)-\gamma b^*(d(t)-d^*).\end{aligned}$$

Then, the time derivative of L along an orbit of system (5) can be derived as

$$\begin{aligned}\frac{dL}{dt}|_{(5)} &= (r(t)-r^*)r'(t)+\eta_1(d(t)-d^*)d'(t)+\eta_2(b(t)-b^*)b'(t)\\ &= [\beta(1-r^*-d^*-b^*)-\beta b^*+\eta_2\phi_b](b(t)-b^*)(r(t)-r^*)\\ &\quad-[\beta(r(t)-r^*)+\eta_2f_b+\eta_2d(t)\gamma](b(t)-b^*)^2+\gamma(\eta_1d^*-\eta_2b^*)(b(t)-b^*)(d(t)-d^*)\\ &\quad-[\beta b(t)+\phi_d+\phi_b](r(t)-r^*)^2+\eta_1(\gamma b(t)-f_d)(d(t)-d^*)^2+(\eta_1\phi_d-\beta b(t))(r(t)-r^*)(d(t)-d^*).\end{aligned}$$

First, we set $\beta(1-r^*-d^*-b^*)-\beta b^*+\eta_2\phi_b=0$, i.e., $\eta_2=\frac{\beta}{\phi_b}(2b^*+r^*+d^*-1)$, which is obviously positive according to the requirement $2b^*+r^*+d^*>1+\phi_br^*/f_b$. On the other hand, we also let $\eta_1d^*-\eta_2b^*=0$, i.e., $\eta_1=\eta_2b^*/d^*>0$. Then, the expression of $\dfrac{dL}{dt}|_{(5)}$ can be reduced as

$$\begin{aligned}\frac{dL}{dt}|_{(5)} &= -[\beta(r(t)-r^*)+\eta_2f_b+\eta_2d(t)\gamma](b(t)-b^*)^2\\ &\quad-\beta b(t)(r(t)-r^*)^2+\eta_1\gamma b(t)(d(t)-d^*)^2-\beta b(t)(r(t)-r^*)(d(t)-d^*)\end{aligned}$$



(a) (b)

Fig. 3. Some solutions of system (5) with parameters given in Example 1. (a) Evolutions of the components $b(t)$, $d(t)$ and $r(t)$ of a specific solution to system (5) with initial vector $(r(0), d(0), b(0)) = (0.4, 0.2, 0.1)$. (b) Plots of some solutions to system (5) with varying γ and different initial values. The stars represent the initial values which are randomly given and the red circle represents the rumor-free equilibrium $E_0 = (0, 0, 0)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{aligned}
&-(\phi_d+\phi_b)(r(t)-r^*)^2-\eta_1 f_d(d(t)-d^*)^2+\eta_1\phi_d(r(t)-r^*)(d(t)-d^*)\\
&=-[\beta(r(t)-r^*)+\eta_2 f_b+\eta_2 d(t)\gamma](b(t)-b^*)^2-\beta b(t)\left[(r(t)-r^*)+\frac{1}{2}(d(t)-d^*)\right]^2\\
&\quad-(\phi_d+\phi_b)\left[(r(t)-r^*)-\frac{\eta_1\phi_d}{2(\phi_b+\phi_d)}(d(t)-d^*)\right]^2+\left[\frac{\eta_1^2\phi_d^2}{4(\phi_b+\phi_d)}-\eta_1 f_d+b(t)\left(\frac{\beta}{4}+\eta_1\gamma\right)\right](d(t)-d^*)^2.
\end{aligned}
$$

By the condition $2b^*+r^*+d^* > 1+\phi_b r^*/f_b$, we have that $\eta_2 f_b-\beta\gamma^* > 0$, leading to $\beta(r(t)-r^*)+\eta_2 f_b+\eta_2 d(t)\gamma > 0$ since $r(t)$, $d(t) \geq 0$. It follows by (11) and $b(t) \in [0, 1]$ that

$$
\frac{\eta_1^2\phi_d^2}{4(\phi_b+\phi_d)}-\eta_1 f_d+b(t)\left(\frac{\beta}{4}+\eta_1\gamma\right) < 0.
$$

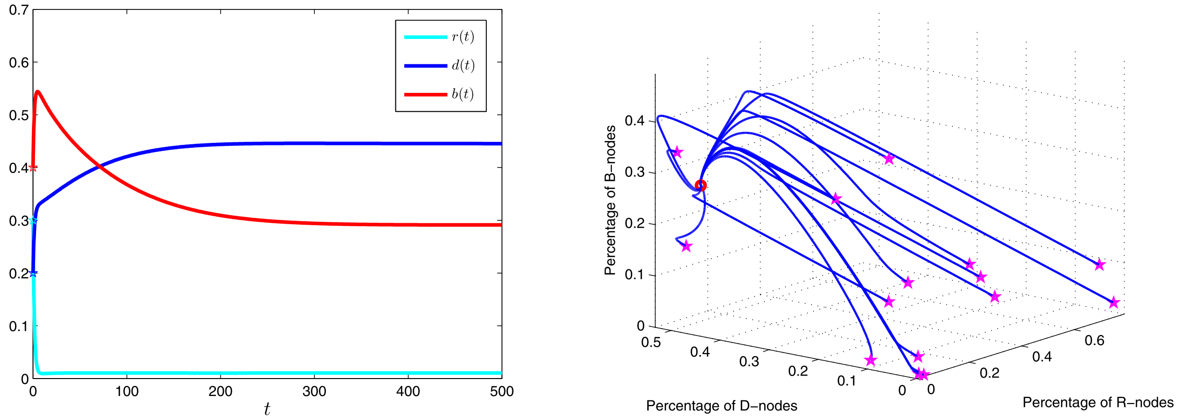
Then, it can be obviously derived that $\dot{L} \leq 0$ holds for all $(r(t), d(t), b(t)) \in \Omega-\{E_0\}$. Moreover, it can be concluded that $\dot{L} = 0$ if and only if $(r(t), d(t), b(t)) = E^*$ with respect to $\Omega-\{E_0\}$. That is, $\dot{L}$ is negative definite. Therefore, E_0 is globally asymptotically stable with respect to $\Omega-\{E_0\}$ under the conditions given above. The proof is complete. □

4. Numerical simulations and analysis

In this section, we are devoted to numerically analyzing the dynamics of model (5). Specifically, some numerical examples are designed to illustrate the above theoretical results. Note that the parameter values in the following simulations are not from real data but are just chosen to illustrate the above results. Note that $\beta = \beta_0\delta N$ and $\gamma = \gamma_0\delta N$, where N represents the network size and the value of δ is affected by the network topology. We consider the spreading of cyber rumor over a given network, i.e., the values of N and δ are properly fixed. For simplicity, we directly specify the values of β and γ in the following simulations. To make the problem physically interesting, the initial values of system (5) are set to be positive, i.e., $r(0) > 0$, $d(0) > 0$ and $b(0) > 0$.

Example 1. Consider system (5) with parameters specified by $f_b = f_d = 0.05$, $\beta = 0.06$, $\gamma = 0.2$, $\phi_b = 0.5$ and $\phi_d = 0.4$. Then, it follows by certain calculations that the threshold $R_0 = 0.6667$, which is below unity, thus Theorem 1 ensures that the rumor-free equilibrium E_0 is globally asymptotically stable.

For system (5) with parameters specified in Example 1 and a set of initial values given by $r(0) = 0.4, d(0) = 0.2, b(0) = 0.1$, Fig. 3a shows the evolutions of $r(t)$, $d(t)$ and $d(t)$, from which it can be observed that the percentages of R-(D-,B-)nodes finally converge to the constant zero. This means that this kind of cyber rumor would finally tend to extinction and is in agreement with Theorem 1. Fig. 3b shows the plot of several solutions of system (5) with different randomly-given initial values and with varying γ. Seven different γ values are fixed by $\gamma = 0.1, \ldots, 0.7$, and for each specific γ value, three sets of randomly-given initial values are considered. It can be seen in Fig. 3b that all of these solutions eventually converge to the rumor-free equilibrium $E_0 = (0, 0, 0)$, which is also consistent with Theorem 1. Fig. 3b also illustrates that varying γ does not impact the global asymptotical stability of E_0 according to Theorem 1.



(a) (b)

Fig. 4. Some solutions of system (5) with parameters given in Example 2. (a) Evolutions of $u(t)$, $b(t)$ and $d(t)$ of a specific solutions of system (5). (b) Plots of tens of solutions to system (5) with randomly-given initial values. The stars represent different initial values, and the red circle represents the rumor equilibrium $E^* = (0.0105, 0.4454, 0.2912)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 2. Consider system (5) with parameters $f_b = f_d = 0.01, \beta = 0.1, \gamma = 0.01, \phi_b = 0.4$ and $\phi_d = 0.3$. Then, it follows by (6) that $R_0 = 5.7143$ (greater than unity).

For system (5) with parameters given in Example 2 and a set of initial conditions satisfying $r(0) = 0.3,\ d(0) = 0.2,\ b(0) = 0.4$, Fig. 4a shows the time evolutions of $r(t)$, $d(t)$, $b(t)$ which approach to the constants 0.0105,0.4454 and 0.2912, respectively. This indicates that in this case cyber rumor will finally keep propagating through the network at a steady level. Fig. 4b shows the plot of its twenty solutions with varying initial values. It can be seen that all of these solutions finally tend to the rumor equilibrium $E^* = (0.0105, 0.4454, 0.2912)$.

From the numerical simulations shown in Fig. 4 along with a number of similar simulation results, we can make a conjecture as follows.

Conjecture 1. *If $R_0 > 1$ and $\Phi < \dfrac{f_b(R_0 - 1)}{f_b R_0 + \gamma}$, then the rumor equilibrium of system (5) is globally asymptotically stable.*

As indicated in the previous sections, the threshold of system (5) plays an important role in determining its dynamics. Thus, in order to effectively prevent and control rumor propagation, it is critical to take certain actions to control the system parameters so that the value of the threshold R_0 is outstandingly below one. In the sequel, some effective measures for achieving this goal are proposed.

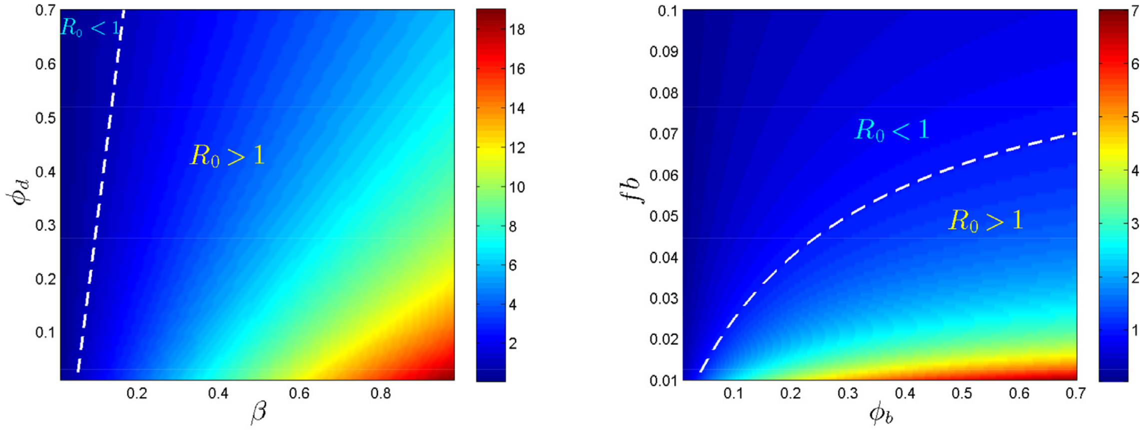
Note that the expression of R_0 involves in the four parameters f_b, ϕ_d, β, ϕ_b of model (5). Furthermore, it can be obviously concluded that an increase of f_b and ϕ_d would result in a corresponding decrease in the value of R_0, respectively. On the other hand, it follows by rewriting (6) as $R_0 = \beta/(f_b(1 + \phi_d/\phi_b))$ that R_0 is strictly increasing with β and ϕ_b, respectively. For our purpose, it is instructive to examine the sensitivity of these parameters on the value of the threshold R_0 (see Fig. 5).

Based on the simulations shown in Fig. 5, we further design some numerical simulations to check whether the inequality $\Phi < f_b(R_0 - 1)/(f_b R_0 + \gamma)$ in Lemma 2 holds for given parameters satisfying $R_0 > 1$. In Fig. 6, the parameters involved in the expression of R_0 are chosen same to those in the simulations of Fig. 5, respectively. Fig. 6a corresponds to Fig. 5a by fixing $f_b = 0.05, \phi_b = 0.3$, and $\gamma = 0.01, f_d = 0.05$, and Fig. 6b corresponds to Fig. 5b by choosing $\beta = 0.1, \phi_d = 0.3$, and $\gamma = f_d = 0.02$. We can observe both in Fig. 6a and b that the inequality $\Phi < f_b(R_0 - 1)/(f_b R_0 + \gamma)$ always holds in the red area where $R_0 > 1$, respectively. Furthermore, the inequality $\Phi < f_b(R_0 - 1)/(f_b R_0 + \gamma)$ also holds for $R_0 > 1$ by checking the varying values of parameter γ in Fig. 6a and b. More simulations like Fig. 6 lead to the same result, and thus we conjecture that the inequality $\Phi < f_b(R_0 - 1)/(f_b R_0 + \gamma)$ always holds for all defined parameters satisfying $R_0 > 1$ (see Remark 5).

In order to control rumor spreading, we need to deeply analyze the parameters of model (5), and further explore how these parameters exactly affect the spreading processes. Clearly, the parameter ϕ_b is introduced to depict the dynamic transitions between the B-nodes and R-nodes. That is, ϕ_b represents the probability of an R-node turning into a B-node.

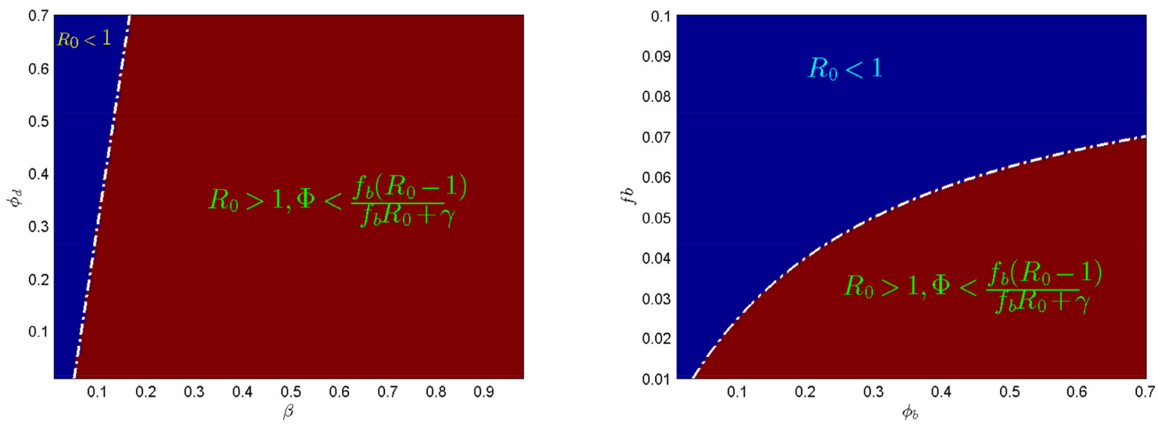
Example 3. Consider system (5) with parameters given by $f_b = f_d = 0.01, \beta = 0.035, \gamma = 0.01, \phi_d = 0.3$. Fig. 7 numerically illustrates the evolutions of the percentage of B-nodes with different ϕ_b and initial values being specified.

Example 4. Consider system (5) with parameters specified by $f_b = f_d = 0.01, \beta = 0.035, \gamma = 0.01, \phi_b = 0.25$. Fig. 8 numerically illustrates the evolutions of $b(t)$ with several values of ϕ_d and initial values being fixed.



(a) (b)

Fig. 5. Illustrations of the impacts of parameters f_b, ϕ_b, β, ϕ_d on the value of R_0. (a) Values of R_0 as a function of varying β, ϕ_d. The other two parameters f_b and ϕ_b are fixed by $f_b = 0.05, \phi_b = 0.3$. The white dashed line highlights $R_0 = 1$ with the parameter values of β and ϕ_d on this line. (b) Values of R_0 as a function of varying f_b, ϕ_b. The other two parameters β and ϕ_d are specified by $\beta = 0.1, \phi_d = 0.3$. The white dashed curve similarly highlights $R_0 = 1$ with the parameter values of f_b and ϕ_b on this curve. In both cases, the white dashed line (curve) separates the whole area into two distinct subareas where $R_0 <$ and $R_0 > 1$, respectively.



(a) (b)

Fig. 6. Illustrations of the inequality $\Phi < f_b(R_0 - 1)/(f_bR_0 + \gamma)$ with parameters satisfying $R_0 > 1$. Both sub-figures are separated by the white line or curve(where $R_0 = 1$) into two areas: the red represents $R_0 > 1$, while the blue represents $R_0 < 1$. (a) It is checked that the inequality $\Phi < f_b(R_0 - 1)/(f_bR_0 + \gamma)$ always holds for parameters $f_b = f_d = 0.05, \phi_b = 0.3, \gamma = 0.01$ and all varying β, ϕ_d values satisfying $R_0 > 1$ (the red area). (b) It is also checked that the inequality $\Phi < f_b(R_0 - 1)/(f_bR_0 + \gamma)$ always holds for parameters $\beta = 0.1, \phi_d = 0.3, \gamma = f_d = 0.02$ and varying f_b, ϕ_b values satisfying $R_0 > 1$ (the red area). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

It can be observed in both Figs. 7 and 8 that all these curves eventually converge to corresponding equilibria, respectively. The final percentage of B-nodes will keep stable at a higher level when the value of the parameter ϕ_b is greater, while the evolutions of $b(t)$ will finally keep steady at a lower level for greater value of ϕ_d. This indicates that for inhibiting cyber rumor spreading we should take suitable measures to increase the value of ϕ_d, and inversely reduce the value of ϕ_b.

The parameter β describes the probability that a U-node converts to be a R-node by a single B-node per unit time. It is essentially determined by the rate of a B-node spreading the rumor message. Thus, it is largely affected by the user behavior. The following example is designed to numerically illustrate the evolutions of $b(t)$ with several different values of β and initial values being fixed.

Example 5. Consider system (5) with parameters specified by $f_b = f_d = 0.01, \gamma = 0.01, \phi_b = \phi_d = 0.5$.

It can be observed in Fig. 9 that the evolutions of $b(t)$ finally tend to corresponding constants, respectively. More specifically, these curves keep stable at a higher value when β values greater. This indicates that reducing the value of β is helpful for controlling the cyber rumor propagation.

Once rumor-believed users realize that the rumor is false, they will no longer believe in it again. Thus, the parameter γ is introduced to depict this situation that a rumor-believed user turns to be a rumor-denied user at a certain probability.

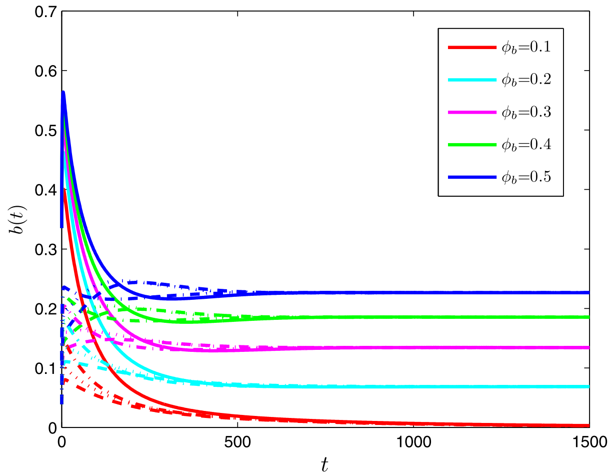


Fig. 7. Evolutions of $b(t)$ in system (5) with varying ϕ_b and other parameters specified in Example 3. Colors represent different values of ϕ_b; linetypes represent evolutions starting from distinct initial values (randomly given).

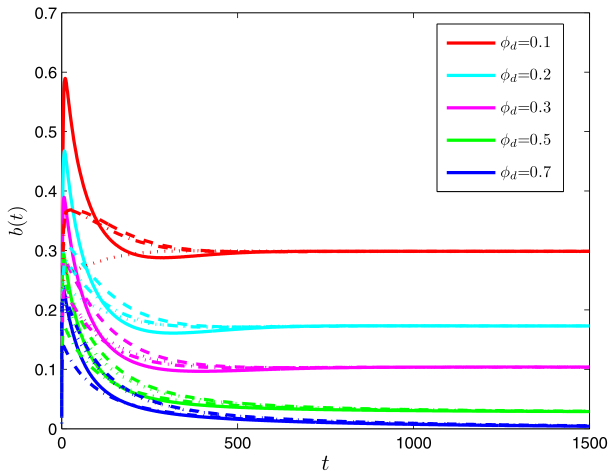


Fig. 8. Evolutions of $b(t)$ in system (5) with varying ϕ_d and other parameters specified in Example 4. Colors represent different values of ϕ_d; linetypes represent evolutions with different initial values (randomly given).

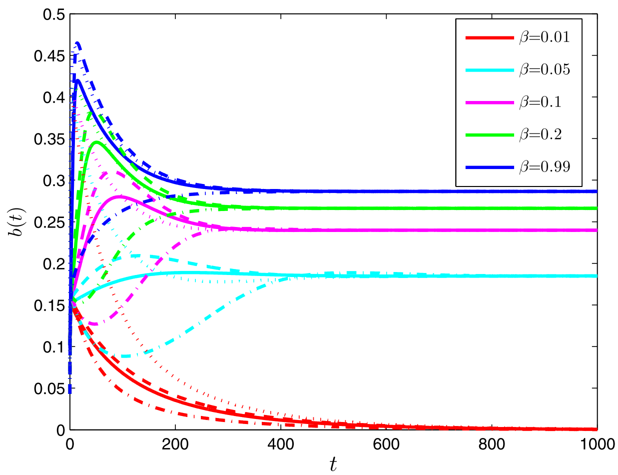


Fig. 9. Evolutions of $b(t)$ in system (5) with varying β and other parameters specified in Example 5. Colors represent different values of β; linetypes represent evolutions with different initial values (randomly given).

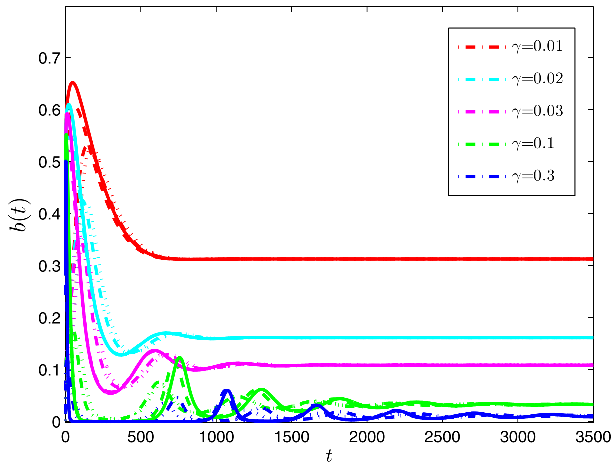


Fig. 10. Evolutions of $b(t)$ in system (5) with varying γ and other parameters specified in Example 6. Colors represent different values of γ; linetypes represent evolutions with different initial values (randomly given).

Example 6. Consider system (5) with parameters specified by $f_b = 0.001$, $f_d = 0.005$, $\beta = 0.05$, $\phi_b = 0.4$, $\phi_d = 0.2$. Then, it follows by (6) that $R_0 = 33.3333$ (greater than unity).

It is shown in Fig. 10 that the evolution processes of $b(t)$ for different cases have some oscillations, but all of them eventually approach to corresponding constants, respectively. This also corresponds to the theoretical results. More specifically, we can also observe that the value of $b(t)$ will finally keep steady at a lower value for greater value of γ. This indicates that the parameter γ actually affects the evolution dynamics of model (5) despite it is not incorporated in the threshold. For the case given in Example 6, the simulations of Fig. 10 suggests that it would be much better to control the value of parameter $\gamma > 0.1$ so as to make the final percentage of B-nodes keep at a slight level.

All the above numerical simulations illustrate the correctness of the theoretical results. Through these simulation results, we can conclude that the spreading of cyber rumor can be effectively inhibited by making suitable measures to control the model parameters.

5. Conclusion

Motivated by the consideration that cyber rumor spreading through Internet has some differences to the traditional rumor spreading by way of mouth-to-mouth, we attempt to develop a novel mathematical model to depict cyber rumor spreading by incorporating its special characteristics. This model is remarkably distinct from previously proposed models, since the mechanism of cyber rumor spreading over social applications is incorporated. Unlike the traditional models where the nodes only represent people among whom the rumors directly spread by mouth, in our model the node is newly defined as a couple of user and its device which considers the interaction of users and devices in the process of cyber rumor spreading. That is, network users mainly derive rumor messages directly from the devices, e.g., through their social network applications, not by the traditional way of mouth-to-mouth. In our new model, the network nodes are newly divided and rumor-denier's behavior of refuting rumors is also addressed. Through the Lyapunov method we get the stability results for the equilibria. Based on the theoretical results and numerical analysis, we propose some suggestions for controlling the cyber rumor spreading.

The model proposed in this paper is fundamental, however, it can be further extended and improved by considering time-varying transmission probabilities and the specific structure of the propagating network. For example, we can further consider that deniers could be changed to believers with some probability due to propaganda efforts. In the current model (4), the spreading network is considered as static and all the parameters are assumed to be constant. However, in the real situation cyber rumors would spread over complex dynamic networks with varying parameters. Toward these directions, our next work is to develop a more realistic model from which we expect to get some more useful suggestions to control cyber rumor spreading. Furthermore, we will also carry out simulations with simulated or even actual data so as to analyze and discuss the significance of our models.

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Appendix A. Calculations of the spreading threshold of system (4).

The authors van den Driessche and Watmough studied how to calculate the spreading thresholds of compartmental models and proposed a general approach [37].

According to this method, we first need to introduce several notations by the physical meaning of this problem. That is, let $\mathcal{F}_i(\mathbf{y})$ be the rate of appearance of new rumor-nodes into compartment i, $\mathcal{V}_i^+(\mathbf{y})$ be the rate of transferring nodes into compartment i by all other means, and $\mathcal{V}_i^-(\mathbf{y})$ be the rate of changing nodes out of compartment i.

Next, in order to conveniently calculate the spreading threshold of the compartmental model (4), we need to denote

$$\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T = (b(t), r(t), u(t), d(t))^T,$$

and then the model (4) can be rewritten as follows

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) = \mathbf{F}(\mathbf{y}) - \mathbf{V}(\mathbf{y}),$$

where $\mathbf{V}(\mathbf{y}) = \mathbf{V}^-(\mathbf{y}) - \mathbf{V}^+(\mathbf{y}), \mathbf{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)^T, \mathbf{V}^- = (\mathcal{V}_1^-, \mathcal{V}_2^-, \mathcal{V}_3^-, \mathcal{V}_4^-)^T, \mathbf{V}^+ = (\mathcal{V}_1^+, \mathcal{V}_2^+, \mathcal{V}_3^+, \mathcal{V}_4^+)^T$, and

$$\mathbf{F}(\mathbf{y}) = \begin{pmatrix} \phi_b y_2 \\ \beta y_1 y_3 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}^-(\mathbf{y}) = \begin{pmatrix} f_b y_1 + \gamma y_1 y_4 \\ (\phi_b + \phi_d) y_2 \\ \beta y_1 y_3 \\ f_d y_4 \end{pmatrix}, \mathbf{V}^+(\mathbf{y}) = \begin{pmatrix} 0 \\ 0 \\ f_d y_4 \\ \gamma y_4 y_1 \end{pmatrix}.$$

Note that $\mathbf{y}_0 = (0, 0, 1, 0)^T$ is the unique rumor-free equilibrium point of the above system which corresponds to the E_0 of system (5). It is easy to verify that all the assumptions of van den Driessche and Watmough [37] are satisfied as follows.

(A1) If $\mathbf{y} \ge 0$, then $\mathcal{F}_i, \mathcal{V}_i^-, \mathcal{V}_i^+ \ge 0$ for $i = 1, 2, 3, 4$.
(A2) If $y_i = 0$, then $\mathcal{V}_i^- = 0$. In particular, if $\mathbf{y} \in Y_2 := \{\mathbf{y} \ge 0 | y_i = 0, i = 1, 2\}$, which is defined as the set of all rumor-free states, then $\mathcal{V}_i^- = 0$ for $i = 1, 2$.
(A3) $\mathcal{F}_i = 0$ if $i > 2$.
(A4) If $\mathbf{y} \in Y_2$, then $\mathcal{F}_i = 0$ and $\mathcal{V}_i^+ = 0$ for $i = 1, 2$.
(A5) If $\mathbf{F}(\mathbf{y})$ is set to zero, then all eigenvalues of $D\mathbf{f}(\mathbf{y}_0)$ have negative real parts.

Then, it follows by Lemma 1 in Ref. [37] that the derivatives $D\mathbf{F}(\mathbf{y}_0)$ and $D\mathbf{V}(\mathbf{y}_0)$ can be partitioned as

$$D\mathbf{F}(\mathbf{y}_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, D\mathbf{V}(\mathbf{y}_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix},$$

where F and V are the 2×2 matrices defined by

$$F = \left[\frac{\partial \mathcal{F}_i}{\partial y_j}(\mathbf{y}_0) \right] \quad \text{and} \quad V = \left[\frac{\partial \mathcal{V}_i}{\partial y_j}(\mathbf{y}_0) \right] \quad \text{with} \quad 1 \le i, j \le 2.$$

That is, $F = \begin{pmatrix} 0 & \phi_b \\ \beta & 0 \end{pmatrix}$, $V = \begin{pmatrix} f_b & 0 \\ 0 & \phi_b + \phi_d \end{pmatrix}$. Note that F is non-negative, V is a non-singular matrix and all eigenvalues of J_4 have positive real parts. Thus, the spectral radius of the matrix FV^{-1} can be computed as

$$\rho(FV^{-1}) = \sqrt{\frac{\phi_b \beta}{f_b(\phi_b + \phi_d)}}.$$

For convenience, in this paper we define the spreading threshold of model (4) as the square of $\rho(FV^{-1})$, i.e.,

$$R_0 = \frac{\phi_b \beta}{f_b(\phi_b + \phi_d)}.$$

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